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13. ABSTRACT (Maximum 200 words)

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The simplification of ignoring any dependence of residuals to attitude errors in their estimation is a good assumption for most terrestrial applications where specific forces are present and may be enhanced through appropriate maneuvers. The application of GPS to exoatmospheric, free falling conditions, however, is an emerging field. Under free falling conditions, however, the specific forces are identically zero and thus the ability to estimate attitude errors through correlation to velocity errors vanishes.

This paper describes the development of an approach to estimate attitude errors in vehicles with changing attitude independent of the value of the specific forces. The paper includes derivation of a more general GPS observation matrix which captures any attitude and attitude rate information that was lost in the conventional derivations and simulation results. The technique developed here has the advantage of being readily implementable in a conventional GPS receiver with only minor software modifications.

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21a. NAME OF RESPONSIBLE INDIVIDUAL	21b. TELEPHONE (include Area Code)	21c. OFFICE SYMBOL
C. A. Bass	(619) 553-6002	Code 804

Attitude Error Estimation with a General Observation Matrix

Charles A. Bass

Naval Command, Control and Ocean Surveillance Center
RDT&E Division

and

Constantine Karmokolias and Azaad Khatri
Galaxy Scientific Corporation

BIOGRAPHY

C. Bass is currently on the staff of the Communications Department of NCCOSC RDT&EDIV San Diego. He received his BS in 1952 in Electrical Engineering from George Washington University.

Dr. Karmokolias is currently Vice President for Engineering at Galaxy Scientific Corporation. He received his BS in 1973 from U of Maine and MS and PhD in 1975 and 1979 from Texas Tech University, all in Electrical Engineering. He is a member of IEEE, AIAA, and ION. He has been involved with development of integrated navigation systems since 1980.

A. Khatri is currently a Senior Engineering Specialist at Galaxy Scientific Corporation. He received his BS in 1986 in Electrical Engineering from Drexel University. He is currently involved with analysis and simulation of advanced integrated navigation systems.

ABSTRACT

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The simplification of ignoring any dependence of residuals to attitude errors in their estimation is a good assumption for most terrestrial applications where specific forces are present and may be enhanced through appropriate maneuvers. The application of GPS to exoatmospheric, free falling conditions, however, is an emerging field. Under free falling conditions, however, the specific forces are identically zero and thus the ability to estimate attitude errors through correlation to velocity errors vanishes.

This paper describes the development of an approach to estimate attitude errors in vehicles with changing attitude independent of the value of the specific forces. The paper includes derivation of a more general GPS observation matrix which captures any attitude and attitude rate information that was lost in the conventional derivations and simulation results. The technique developed here has the advantage of being readily implementable in a conventional GPS receiver with only minor software modifications.

1.0 INTRODUCTION AND BACKGROUND

In integrated GPS/INS navigation systems the observation matrix is conventionally derived under the assumption of coincidence of the antenna phase center and the reference point. Consequently, the pseudo range (PR) and delta range (DR) measurement residuals are modeled to have coupling only to position, velocity, user clock bias and drift error states. Potential coupling to attitude errors or attitude rate errors is ignored. Instead, residuals are simply corrected for the antenna lever arm. The needed attitude information for antenna lever arm correction is generally provided by estimation of attitude errors and propagation of the corrected attitude "total state" (typically a direction cosine matrix or quaternion) to the time(s) of measurement. Attitude error estimation for such an implementation depends entirely on correlation between the attitude error and the velocity error. Specifically, contribution of the attitude error to the velocity error is due to the product of the attitude error vector by the skew-symmetric matrix of specific forces within the dynamics F matrix.

This approach has been successfully applied to many applications such as terrestrial navigation. There are, however, applications where the correlation between attitude error and velocity error vanishes. For example, when an exoatmospheric host vehicle is free-falling, the

specific forces are zero for all practical purposes. Then, although GPS may succeed in estimating position and velocity errors fairly well, attitude estimation is not possible. Moreover, the correction for the antenna lever arm becomes flawed as attitude errors grow due to instrumentation errors such as gyro bias and scale factor errors.

Typically, GPS-based attitude error estimation methods rely on receiving carriers from multiple GPS satellites and processing the single difference of their phases. A minimum of 3 non-collinear antennas is required in most methods as well as considerable receiver hardware design features and/or special software processing beyond the typical GPS/INS designs.

An alternative approach to attitude estimation is presented in this paper. We seek to exploit the attitude information inherently present in the PR and the DR measurements obtained with a single offset GPS antenna mounted on a platform undergoing attitude changes. The attitude information is primarily recovered through processing DR measurements. The PR measurement is less useful because of the generally high code loop noise. The implementation of this approach does not require any special receiver hardware and only a few extra terms in the standard GPS/INS Kalman filter observation matrix H. For "good" performance, an integrated GPS/INS system is required. The INS provides for propagation of the attitude estimates in between GPS updates. In some specific applications, however, GPS alone may suffice.

An overview of the contents of this paper is as follows. Sections 2, 3, and 4 contain independent mathematical developments of this approach to attitude estimation. Section 2 provides a general mathematical development of the observation matrix H which is applicable to any changing host vehicle attitude. The H matrix is developed without reference to any of the other models that are part of a GPS/INS Kalman Filter. Section 3 provides a mathematical development of the observation matrix H which is also generally applicable to any changing host vehicle attitude. This development, however, is based on a state space formulation and as a consequence it employs and thereby relies on prior developments -specifically, the state transition matrix and the plant noise model. Section 3 also provides a quick comparison of this method to the more familiar "interferometric" GPS-based attitude determination methods. Section 4 provides a simpler analysis of attitude estimation but limited to a spinning body scenario. As a result it facilitates gaining certain insights to and identifying sensitivities in spinning body applications. Section 5 provides simulation results for a spinning body application using, however, the more general formulation of Section 3. Section 6 provides a summary.

2.0 GENERAL DERIVATION OF THE OBSERVATION MATRIX H

This section provides a general mathematical development of the observation matrix H which is applicable to any changing host vehicle attitude. We first derive H matrix row vectors corresponding to PR residuals for a single "offset" GPS antenna and then proceeds to derive H matrix row vectors for DR residuals. It is recognized that for limitations in the antenna lever arm length, the code loop noise will generally overwhelm the relation of PR residuals to attitude error. The DR residuals, however, being based on the very low noise carrier tracking may well afford estimation of attitude and attitude rate errors.

It is assumed that the INS can measure the antenna rotation with sufficient accuracy so that the unwanted phase shifts due to the antenna rotation as well as other antenna phase imperfections can be compensated. The problems and issues associated with the corresponding compensation of DR residuals and antenna calibrations were not investigated here.

Pseudo Range Measurement

The geometry of the problem is shown in Figure 1. With respect to some convenient coordinate frame (e.g., Earth Centered Earth Fixed - ECEF) designated as Σ_a, the PR between the antenna phase center and the jth satellite is given by

$$z_j = \underline{e}_j^T (\underline{r}_u + \underline{r}_A - \underline{r}_{sj}) + B_u + v_{PRj} \qquad (1)$$

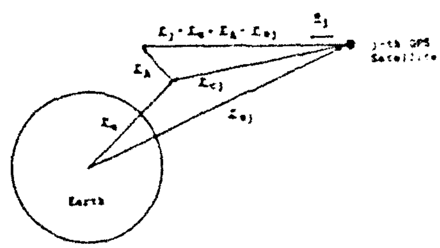

Figure 1. Problem Geometry

where v_{PRj} represents measurement noise. As bias in the satellite clock is not modeled, it is not included in (1). We will suppress any designation for our selected convenient reference frame Σ_a but maintain superscripts for other specially designated frames. $\underline{r}_A$ is the random process vector providing the location of the GPS antenna but as represented in Σ_a, $\underline{r}_A$ will be rapidly changing due to the changing attitude of the host vehicle. However, when represented in the host vehicle body frame Σ_b (as $\underline{r}^b_A$), it will be constant, well known, and related to $\underline{r}_A$ through the direction cosine matrix $\underline{\underline{C}}_b^a$ which is modeled as a random process

$$\underline{r}_A(t) = \underline{\underline{C}}_b^a(t)\, \underline{r}_A^b \qquad (2)$$

In the approach of this section it is convenient to disregard PR and DR measurement noise. The spatial differential which we will take with respect to the nominal or predicted value (i.e. evaluated along the nominal trajectory) corresponds to the PR measurement residual. That is,

$$\begin{aligned}\delta z_j &= \delta[\underline{e}_j^T(\underline{r}_u+\underline{r}_A-\underline{r}_{sj})] + \delta B_u \\ &= (\underline{r}_{un}+\underline{r}_{An}-\underline{r}_{sjn})^T\delta\underline{e}_j + \underline{e}_{jn}^T\delta(\underline{r}_u+\underline{r}_A-\underline{r}_{sj}) + \delta B_u\end{aligned} \tag{3}$$

As $\underline{e}_j$ is of constant (unit) magnitude, $\delta\underline{e}_j$ for small magnitudes must be orthogonal to $\underline{e}_j$ and, therefore, to $\underline{r}_u + \underline{r}_A - \underline{r}_{sj}$ to which $\underline{e}_j$ is parallel. Hence,

$$\begin{aligned}\delta z_j &= \underline{e}_{jn}^T\delta\underline{r}_u + \underline{e}_{jn}^T\delta\underline{r}_A + \delta B_u \\ &= \underline{e}_{jn}^T\delta\underline{r}_u + \underline{e}_{jn}^T(\delta\underline{\underline{C}}_b^e)\,\underline{r}_{An}^{bn} + \delta B_u\end{aligned} \tag{4}$$

as $\underline{r}_{An}^{bn} = \underline{r}_A^b$ is a constant (column) vector. From the development in [1] we have

$$\delta z_j = \underline{e}_{jn}^T\delta\underline{r}_u - \underline{e}_{jn}^T\underline{\underline{R}}_{An}\,\underline{\delta\phi} + \delta B_u \tag{5}$$

where $\underline{\underline{R}}_{An}$ is the skew-symmetric matrix form of vector $\underline{r}_{An}$

$$\underline{\underline{R}}_{An} = \begin{bmatrix} 0 & -r_{Azn} & r_{Ayn} \\ r_{Azn} & 0 & -r_{Axn} \\ -r_{Ayn} & r_{Axn} & 0 \end{bmatrix} \tag{6}$$

The H matrix for PR measurements only but referred to the GPS antenna phase center is given by

$$\begin{array}{c} \\ \delta z_1 \\ \delta z_2 \\ \vdots \\ \delta z_k \end{array}
\begin{array}{c} \begin{array}{ccc} \delta\underline{r}_u & \underline{\delta\phi} & \delta B_u \end{array} \\ \begin{bmatrix} \underline{e}_{1n}^T & -\underline{e}_{1n}^T\underline{\underline{R}}_{An} & 1 \\ \underline{e}_{2n}^T & -\underline{e}_{2n}^T\underline{\underline{R}}_{An} & 1 \\ \vdots & \vdots & \vdots \\ \underline{e}_{kn}^T & -\underline{e}_{kn}^T\underline{\underline{R}}_{An} & 1 \end{bmatrix} \end{array} \tag{7}$$

If, instead of bringing the H matrix to the antenna phase center, had we derived an H matrix based on the antenna phase center coinciding with the reference point, that is $\underline{\underline{R}}_A = \underline{\underline{O}}$ (but still using measurement residuals based on antenna lever arm corrections), the resulting H matrix would show no coupling between the residuals and $\underline{\delta\phi}$. By eliminating then the $\underline{\delta\phi}$ column in (7) we readily obtain the usual H matrix encountered when PR measurements are presented to a Kalman filter implemented in ECEF. The more general H matrix given in (7) appears to have promise for attitude observability but would require a large offset distance of the antenna phase center from the reference point of estimated quantities to overcome the noise in the GPS receiver code loops.

Delta Range Measurements

For a DR measurements model, we seek the spatial difference process of a temporal difference process. We will maintain a first order approximation to the spatial and temporal difference parts in order to use a linear Kalman filter. Taking the time differential of the PR measurement residual we have

$$d(\delta z_j) = d(\underline{e}_{jn}^T\delta\underline{r}_u) - d(\underline{e}_{jn}^T\underline{\underline{R}}_{An}\underline{\delta\phi}) + d(\delta B_u) \tag{8}$$

and on expanding

$$\begin{aligned}d(\delta z_j) = {} & (d\underline{e}_{jn}^T)\delta\underline{r}_u + \underline{e}_{jn}^T d(\delta\underline{r}_u) \\ & -(d\underline{e}_{jn}^T)\underline{\underline{R}}_{An}\underline{\delta\phi} - \underline{e}_{jn}^T(d\underline{\underline{R}}_{An})\underline{\delta\phi} - \underline{e}_{jn}^T\underline{\underline{R}}_{An}d(\underline{\delta\phi}) + d(\delta B_u)\end{aligned} \tag{9}$$

By linearity we can exchange $d(\cdot)$ and $\delta(\cdot)$ for all except $\underline{\delta\phi}$ which is not separable with any proper meaning. Since the rate of attitude error is equal to the angular velocity error

$$D(\underline{\delta\phi}) = \underline{\omega}_{eb} - \underline{\omega}_{e,bn} = \delta\underline{\omega}_{eb} \tag{10}$$

$$d(\underline{\delta\phi}) = \delta\underline{\omega}_{eb}dt$$

Since

$$\delta\underline{\omega}_{eb} = \underline{\omega}_{eb} - \underline{\omega}_{e,bn} = (\underline{\omega}_{ei} + \underline{\omega}_{ib}) - (\underline{\omega}_{ei} + \underline{\omega}_{i,bn})$$

$$\delta\underline{\omega}_{eb} = \underline{\omega}_{ib} - \underline{\omega}_{i,bn} = \delta\underline{\omega}_{ib} \tag{11}$$

we obtain from (9) and (11)

$$\begin{aligned}\delta(dz_j) = {} & (d\underline{e}_{jn}^T)\delta\underline{r}_u + \underline{e}_{jn}^T(\delta\underline{v}_u)dt - (d\underline{e}_{jn}^T)\underline{\underline{R}}_{An}\underline{\delta\phi} - \underline{e}_{jn}^T(d\underline{\underline{R}}_{An})\underline{\delta\phi} \\ & -\underline{e}_{jn}^T\underline{\underline{R}}_{An}(\underline{\delta\omega}_{ib})dt + \delta f_u dt\end{aligned} \tag{12}$$

where we have used

$$d\underline{r}_u = \frac{\partial\underline{r}_u}{\partial t}dt = \underline{v}_u dt \text{ and } dB_u = \frac{\partial B_u}{\partial t}dt$$

Integrating from t_i-T to t_i under the assumption of temporally "well behaved" $\delta\underline{r}_u$, $\underline{\delta\phi}$, $\delta\underline{v}_u$, $\delta\underline{\omega}_{ib}$, and δf_u and using the Theorem of Mean Value for integration we obtain (see [1] for details)

$$\begin{aligned}\delta(\Delta z_j(t_i)) = {} & \Delta\underline{e}_{jn}^T(t_i)\,\delta\underline{r}_u(t_i) + \underline{e}_{jn}^T(t_a)\,T\delta\underline{v}_u \\ & -[\Delta\underline{e}_{jn}^T(t_i)\underline{\underline{R}}_{An}(t_b) + \underline{e}_{jn}^T(t_c)\Delta\underline{\underline{R}}_{An}(t_i)]\,\underline{\delta\phi}(t_i) \\ & -\underline{e}_{jn}^T(t_d)\,TR_{An}(t_e)\,\delta\underline{\omega}_{ib}(t_i) + T\delta f_u(t_i)\end{aligned} \tag{13}$$

where $t_i - T \le t_a, t_b, t_c, t_d, t_e < t_i$

The weakness of this method lies in the inability to precisely determine the appropriate values for these times in the interval (t_i, t_i-T). Incorporating both PR and DR residuals, the resulting H matrix is

$$\begin{array}{c} \\ \delta z_1 \\ \delta z_2 \\ \vdots \\ \delta z_k \\ \delta(\Delta z_1) \\ \delta(\Delta z_2) \\ \vdots \\ \delta(\Delta z_k) \end{array}
\begin{array}{c} \begin{array}{cccccc} \delta\underline{r}_u & \delta\underline{v}_u & \underline{\delta\phi} & \delta\underline{\omega}_{ib} & \delta B_u & \delta f_u \end{array} \\ \begin{bmatrix} \underline{e}_{1n}^T & \underline{0}^T & -\underline{e}_{1n}^T\underline{\underline{R}}_{An} & \underline{0}^T & 1 & 0 \\ \underline{e}_{2n}^T & \underline{0}^T & -\underline{e}_{2n}^T\underline{\underline{R}}_{An} & \underline{0}^T & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \underline{e}_{kn}^T & \underline{0}^T & -\underline{e}_{kn}^T\underline{\underline{R}}_{An} & \underline{0}^T & 1 & 0 \\ \Delta\underline{e}_{1n}^T & \underline{e}_{1n}^T T & -(\Delta\underline{e}_{1n}^T\underline{\underline{R}}_{An} + \underline{e}_{1n}^T\Delta\underline{\underline{R}}_{An}) & -\underline{e}_{1n}^T T\underline{\underline{R}}_{An} & 0 & T \\ \Delta\underline{e}_{2n}^T & \underline{e}_{2n}^T T & -(\Delta\underline{e}_{2n}^T\underline{\underline{R}}_{An} + \underline{e}_{2n}^T\Delta\underline{\underline{R}}_{An}) & -\underline{e}_{2n}^T T\underline{\underline{R}}_{An} & 0 & T \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \Delta\underline{e}_{kn}^T & \underline{e}_{kn}^T T & -(\Delta\underline{e}_{kn}^T\underline{\underline{R}}_{An} + \underline{e}_{kn}^T\Delta\underline{\underline{R}}_{An}) & -\underline{e}_{kn}^T T\underline{\underline{R}}_{An} & 0 & T \end{bmatrix} \end{array} \tag{14}$$

If, instead of bringing the H matrix to the antenna phase center, had we derived an H matrix based on the antenna phase center coinciding with the reference point, that is $\underline{\underline{R}}_A = \underline{\underline{O}}$ (but still using measurement residuals based on antenna lever arm corrections), the resulting H matrix would show no coupling between the residuals $\underline{\delta\phi}$ and $\delta\underline{\omega}_{ib}$. By eliminating the $\underline{\delta\phi}$ column in (25) we obtain the usual H matrix encountered when both PR and DR measurements are presented to a Kalman filter implemented in ECEF.

When Σ_A is a local level frame Σ_n, we generally want the terrestrial velocity of the user rather than the velocity as perceived by an observer fixed to Σ_b. Using terrestrial velocity $\underline{v}$, the observation matrix H becomes

$$
\begin{array}{c}
 \\ \delta z_1 \\ \delta z_2 \\ \vdots \\ \delta z_k \\ \delta(\Delta z_1) \\ \delta(\Delta z_2) \\ \vdots \\ \delta(\Delta z_k)
\end{array}
\begin{array}{c}
\begin{array}{cccccc} \delta\underline{r}_A & \delta\underline{v} & \underline{\delta\phi} & \delta\underline{\omega}_{ib} & \delta B_u & \delta\dot{c}_u \end{array} \\
\left[\begin{array}{cccccc}
\underline{e}_{1n}^T & \underline{0}^T & -\underline{e}_{1n}^T\underline{\underline{R}}_A & \underline{0}^T & 1 & 0 \\
\underline{e}_{2n}^T & \underline{0}^T & -\underline{e}_{2n}^T\underline{\underline{R}}_A & \underline{0}^T & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\underline{e}_{kn}^T & \underline{0}^T & -\underline{e}_{kn}^T\underline{\underline{R}}_A & \underline{0}^T & 1 & 0 \\
(\underline{\underline{C}}_e^n\Delta\underline{e}_{1n}^e)^T & \underline{e}_{1n}^T T & -(\Delta\underline{e}_1^T\underline{\underline{R}}_{An} + \underline{e}_{1n}^T\Delta\underline{\underline{R}}_{An}) & -\underline{e}_{1n}^T T\underline{\underline{R}}_{An} & 1 & T \\
(\underline{\underline{C}}_e^n\Delta\underline{e}_{2n}^e)^T & \underline{e}_{2n}^T T & -(\Delta\underline{e}_2^T\underline{\underline{R}}_{An} + \underline{e}_{2n}^T\Delta\underline{\underline{R}}_{An}) & -\underline{e}_{2n}^T T\underline{\underline{R}}_{An} & 0 & T \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(\underline{\underline{C}}_e^n\Delta\underline{e}_{kn}^e)^T & \underline{e}_{kn}^T T & -(\Delta\underline{e}_k^T\underline{\underline{R}}_{An} + \underline{e}_{kn}^T\Delta\underline{\underline{R}}_{An}) & -\underline{e}_{kn}^T T\underline{\underline{R}}_{An} & 0 & T
\end{array}\right]
\end{array}
\tag{15}
$$

We note that $\delta\underline{\omega}_{ib}$ includes the effects of gyro bias $\delta\underline{\omega}_B$, gyro scale factor error, etc.

3.0 STATE SPACE FORMULATION

In developing a measurement model, the method presented in section 2 has taken a direct approach. The strength of such an approach is that it has remained independent of any other analysis and thereby has been exploratory in seeking coupling between PR and DR measurement residuals and the error states. The results for the PR residuals need no further elaboration and are useful as derived. With regard to DR residuals, however, a weakness is that it has not been established where certain terms in (13) need to be evaluated. For example, when the time interval T over which the temporal difference is taken is small such that the change in $\underline{\underline{R}}$ is small, the problem is minimal and any reasonable choice of evaluation time will provide useful results. For greater efficiency in coupling to the attitude errors, however, we wish to maximize $\Delta\underline{\underline{R}}_A$. The choice of where to best evaluate $\underline{\underline{R}}_A$ in the interval $(t_i\text{-}T, t_i)$ is uncertain.

In this section we utilize a presumed existing dynamic model to develop the temporal differences required for the relations through backward transition matrices. The strength of this approach is that there is no uncertainty as to where to evaluate the terms in the resulting relations. The weakness, however, is that it depends on a previously developed dynamics model and is subject to any inadequacies in this model. In particular, uncertainty in the backward time propagation contributes to uncertainty in the measurement model.

The total state dynamic model is taken to be continuous and given by

$$\dot{\underline{x}}(t) = \underline{f}[\underline{x}(t), t] + \underline{\underline{G}}(t)\underline{w}(t) \tag{16}$$

where $\underline{w}(t)$ is a zero-mean white Gaussian noise process. For the purpose of either a linearized or extended Kalman filter, we assume a nominal trajectory of similar form

$$\dot{\underline{x}}_n(t) = \underline{f}[\underline{x}_n(t), t] \tag{17}$$

The spatial difference process

$$\delta\underline{x}(t) = \underline{x}(t) - \underline{x}_n(t) \tag{18}$$

has the first order approximation

$$\delta\dot{\underline{x}}(t) = \underline{\underline{F}}[t;\underline{x}_n(t)]\delta\underline{x}(t) + \underline{\underline{G}}(t)\underline{w}(t) \tag{19}$$

where

$$\underline{\underline{F}}[t;\underline{x}_n(t)] \triangleq \frac{\partial\underline{f}(\underline{x},t)}{\partial\underline{x}}\bigg|_{\underline{x}=\underline{x}_n(t)}$$

The discrete time measurements are taken to be a nonlinear function of the state with an additive white noise sequence.

$$\underline{z}(t_i) = \underline{h}[\underline{x}(t_i), t_i] + \underline{v}(t_i) \tag{20}$$

Associated with the nominal trajectory is the sequence of nominal measurements

$$\underline{z}_n(t_i) = \underline{h}[\underline{x}_n(t_i), t_i] \tag{21}$$

The spatial difference process

$$\delta\underline{z}(t_i) = \underline{z}(t_i) - \underline{z}_n(t_i) = \underline{h}[\underline{x}(t_i), t_i] - \underline{h}[\underline{x}_n(t_i), t_i] + \underline{v}(t_i) \tag{22}$$

has the first order approximation

$$\delta\underline{z}(t_i) = \underline{\underline{H}}[t_i;\underline{x}_n(t_i)]\delta\underline{x}(t_i) + \underline{v}(t_i) \tag{23}$$

where

$$\underline{\underline{H}}[t_i;\underline{x}_n(t_i)] \triangleq \frac{\partial\underline{h}[\underline{x}(t_i), t_i]}{\partial\underline{x}}\bigg|_{\underline{x}=\underline{x}_n(t_i)} \tag{24}$$

As a discrete time process, the measured DR corrupted by measurement noise is

$$z_{DRj}(t_i) = R_j(t_i) - R_j(t_i - T) + v_{DRj}(t_i) \tag{25}$$

where

$$R_j(t_k) = \|\underline{r}_j(t_k)\| + B_u(t_k)$$

The nominal DR measurement corresponding to the nominal trajectory is

$$z_{DRjn}(t_i) = R_{jn}(t_i) - R_{jn}(t_i - T) \tag{26}$$

The DR measurement residual discrete time process is then

$$\begin{aligned}\delta z_{DRj}(t_i) &= z_{DRj}(t_i) - z_{DRjn}(t_i) \\ &= R_j(t_i) - R_{jn}(t_i) - (R_j(t_i - T) - R_{jn}(t_i - T)) + v_{DRj}(t_i)\end{aligned} \tag{27}$$

or from Figure 1

$$\begin{aligned}\delta z_{DRj}(t_i) = &\,|\underline{r}_j(t_i)| - |\underline{r}_j(t_i) - \delta\underline{r}_u(t_i) - \delta\underline{r}_A(t_i)| + \delta B_u(t_i) \\ &-\{|\underline{r}_j(t_i - T)| - |\underline{r}_j(t_i - T) - \delta\underline{r}_u(t_i - T) - \delta\underline{r}_A(t_i - T)| + \delta B_u(t_i - T)\} \\ &+ v_{DRj}(t_i)\end{aligned} \tag{28}$$

Using the skew-symmetric form $\underline{\underline{R}}_{An}(t)$ for vector $\underline{r}_{An}(t)$, we have

$$\delta\underline{r}_A(t) = -\underline{\underline{R}}_{An}(t)\,\underline{\delta\phi}(t) \tag{29}$$

We then have for the discrete time measurement residual process

$$\begin{aligned}\delta z_{DRj}(t_i) = &\,|\underline{r}_j(t_i)| - |\underline{r}_j(t_i) - \delta\underline{r}_u(t_i) + \underline{\underline{R}}_{An}(t_i)\underline{\delta\phi}(t_i)| + \delta B_u(t_i) \\ &-\{|\underline{r}_j(t_i - T)| - |\underline{r}_j(t_i - T) - \delta\underline{r}_u(t_i - T) + \underline{\underline{R}}_{An}(t_i - T)\underline{\delta\phi}(t_i - T)| + \delta B_u(t_i - T)\} \\ &+ v_{DRj}(t_i)\end{aligned} \tag{30}$$

We define 3 x n matrices $\underline{\underline{K}}_p$ and $\underline{\underline{K}}_\phi$ and row vector (1 x n matrix) $\underline{k}_B^T$ such that we can relate to the full error state vector at measurement time t_i.

$$\begin{aligned}\delta\underline{r}_u(t_i) &= \underline{\underline{K}}_p\,\delta\underline{x}(t_i) \\ \underline{\delta\phi}(t_i) &= \underline{\underline{K}}_\phi\,\delta\underline{x}(t_i) \\ \delta B_u(t_i) &= \underline{k}_B^T\,\delta\underline{x}(t_i)\end{aligned} \tag{31}$$

At measurement time t_i-T we obtain the corresponding quantities but using the backward transition matrix

$$\begin{aligned}\delta\underline{r}_u(t_i - T) &= \underline{\underline{K}}_p[\Phi(t_i - T, t_i)\,\delta\underline{x}(t_i) + \underline{w}_D(t_i - T)] \\ \underline{\delta\phi}(t_i - T) &= \underline{\underline{K}}_\phi[\Phi(t_i - T, t_i)\,\delta\underline{x}(t_i) + \underline{w}_D(t_i - T)] \\ \delta B_u(t_i - T) &= \underline{k}_B^T[\Phi(t_i - T, t_i)\,\delta\underline{x}(t_i) + \underline{w}_D(t_i - T)]\end{aligned} \tag{32}$$

where $\underline{w}_D(t_i\text{-}T)$ is the driven response at t_i-T due to the presence of the white noise in (16) during the backward interval from t_i to t_i-T. $\underline{w}_D(t_i\text{-}T)$, by virtue of the white noise in the continuous model, is a white noise sequence. It is here that uncertainty in the dynamics model is introduced in the measurement model.

The discrete measurement residual process is now

$$\begin{aligned}\delta z_{DRj}(t_i) = &\,|\underline{r}_j(t_i)| - |\underline{r}_j(t_i) - [\underline{\underline{K}}_p - \underline{\underline{R}}_{An}(t_i)\underline{\underline{K}}_\phi]\delta\underline{x}(t_i)| \\ &-\{|\underline{r}_j(t_i - T)| - |\underline{r}_j(t_i - T) - [\underline{\underline{K}}_p - \underline{\underline{R}}_{An}(t_i - T)\underline{\underline{K}}_\phi][\underline{\underline{\Phi}}(t_i - T, t_i)\,\delta\underline{x}(t_i) + \underline{w}_D(t_i - T)]|\} \\ &+\underline{k}_B^T\{[\underline{\underline{I}} - \Phi(t_i - T, t_i)]\delta\underline{x}(t_i) + \underline{w}_D(t_i - T)\} + v_{DRj}(t_i)\end{aligned} \tag{33}$$

If we approximate the backward transition matrix by the first order form

$$\underline{\underline{\Phi}}(t_i - T, t_i) = \underline{\underline{I}}_{nxn} - \underline{\underline{F}}[t_i;\underline{x}_n(t_i)]T \tag{34}$$

we obtain the jth row of H corresponding to the measurement residual for the jth satellite as

$$\begin{aligned}\underline{h}_j^T(t_i;\underline{x}_n(t_i)) = &\,(\underline{e}_{jn}^T(t_i) - \underline{e}_{jn}^T(t_i - T))\underline{\underline{K}}_p \\ &-(\underline{e}_{jn}^T(t_i)\underline{\underline{R}}_{An}(t_i) - \underline{e}_{jn}^T(t_i - T)\underline{\underline{R}}_{An}(t_i - T))\underline{\underline{K}}_\phi \\ &+\underline{e}_{jn}^T(t_i - T)\underline{\underline{K}}_p\underline{\underline{F}}[t_i;\underline{x}_n(t_i)]T \\ &-\underline{e}_{jn}^T(t_i - T)\underline{\underline{R}}_{An}(t_i - T)\underline{\underline{K}}_\phi\underline{\underline{F}}[t_i;\underline{x}_n(t_i)]T \\ &+\underline{k}_B^T\underline{\underline{F}}[t_i;\underline{x}_n(t_i)]T\end{aligned} \tag{35}$$

The DR measurement results of this method were compared to those of the first method in detail in [1]. The results were shown to be identical other than the time of evaluation is quite clear for the second method. This overcomes the major weakness in the first approach. The second method, as previously noted, increases uncertainty in the measurement model for any inadequacies in the dynamics model used. Both methods may be extended beyond a first order model.

Comparison To Standard Attitude Measurement Techniques

Although a comparison to an exhaustive list of other GPS-based attitude determination methods is beyond the scope of this report, a quick comparison of the method described in this report to "interferometric" attitude determination methods (see [2] and the references cited there) may be useful at this point. We can construct a configuration which affords a convenient general comparison between the technique presented here and the more familiar attitude measurement techniques. For purposes of comparison we take any two non-coincident GPS antennas of the "standard" interferometric method and the set of their phase differences for each of four GPS satellites to comprise a *spatial* GPS Interferometer. A set of DR measurements taken over the same time interval for four GPS satellites using a single antenna in effect provides a *temporal* version of a GPS Interferometer as defined above. While more restrictive than need be due to use of recursive estimation techniques which preserve information by propagating it forward, the typical implementation of a conventional five channel GPS receiver having PR and DR measurements would in fact conform to such a configuration.

A brief comparison between the technique described in this report (denoted as "DR-Based method") and standard GPS interferometric methods is provided in Table 1.

4.0 SIMPLIFIED DR ANALYSIS FOR A SPINNING BODY

In this section we investigate the observability afforded to attitude errors by the DR measurement specifically for a spinning body scenario. Restricting the analysis to a simple spinning body scenario facilitates gaining certain insights to and identifying sensitivities in spinning body applications. The analysis is limited to the

Table 1 Comparison of Standard Interferometric GPS-based attitude measurement techniques to DR-Based method

Standard (Multiple Antennas)	DR-Based (Single Antenna)
Requires multiple antennas and a special GPS receiver design. No requirement on host vehicle motion relative to GPS satellite constellation.	Requires only a single antenna and a standard GPS receiver having PR and DR measurement capability. Requires changing attitude of host vehicle relative to GPS satellite constellation. Integrated GPS/INS generally needed for good performance.
Any two non-coincident antennas receiving carriers of four GPS satellites and the single difference of their phases constitute a *spatial* GPS Interferometer.	Each set of DR measurements for four GPS satellites using a single antenna on host vehicle undergoing attitude changes is equivalent to constructing *temporally* a GPS Interferometer having as antennas the start and stop antenna locations. The accumulated phases of the DR measurement set are equivalent to the set of differences of received phases at the start and stop antenna locations.
A minimum of two non-parallel GPS Interferometers (a minimum of three non-collinear antennas) is required for attitude determination at any time.	A minimum of two non-parallel equivalent GPS interferometers (two DR measurement sets) required for attitude determination with INS propagation of information over the DR time period. Attitude is determined for end of period and is propagated forward by INS.
The satellite phase errors are removed by the single *spatial* difference in phase between the two antennas of the GPS Interferometer.	Removal of satellite phase errors is inherent to each DR measurement.
Ambiguity resolution for integer cycles must be accomplished for the single phase differences. Use of redundancy, search procedures, triple differences, etc., are required.	No ambiguities in each DR measurement provided no loss of carrier lock or cycle slip in GPS receiver channel associated with GPS satellite.
May require calibration and compensation for antenna induced error due to phase pattern differences between the two antennas constituting the GPS interferometer.	Fundamental phase change due to attitude changes in single antenna must be compensated for. Antenna must also be calibrated for phase pattern anomalies.
Double difference (difference between two independent single differences) required to remove receiver timing errors, electrical path bias errors, etc.	Use of single antenna and standard DR measurement obviate the need for double differences.
Direct solution of attitude performed.	Recursive estimation (using Kalman Filter) of attitude errors relative to nominal trajectory established by INS and its corrections (resets) is performed.

case of an initial (constant) attitude error in a rotating but not translating vehicle.

From (15), the attitude error $\underline{\delta\phi}$ is related to the DR residual for the jth satellite via

$$\underline{H}_{j\delta\phi} = -(\Delta\underline{e}_{jn}^{T}\underline{R}_{An} + \underline{e}_{jn}^{T}\Delta\underline{R}_{An}) \tag{36}$$

For our choice of reference frame and short periods of time the change in the unit vector $\underline{e}_{jn}$ is small so, we can assume that $\Delta\underline{e}_{jn}$ is approximately zero. Then (36) reduces to

$$\underline{H}_{j\delta\phi} = -\underline{e}_{jn}^{T}\Delta\underline{R}_{An} \tag{37}$$

The contribution of the attitude error $\underline{\delta\phi}$ to the DR measurement residual is obtained by multiplying both sides of (36) by $\underline{\delta\phi}$.

$$H_{j\delta\phi}\underline{\delta\phi} = \underline{e}_{jn}^{T}\Delta\underline{R}_{An}\underline{\delta\phi} \tag{38}$$

From (38) we conclude that the contribution of the attitude error $\underline{\delta\phi}$ to the DR residual is maximum when the vectors $\underline{e}_{jn}$, $\Delta\underline{r}_{A}$, and $\underline{\delta\phi}$ are mutually orthogonal. The contribution is zero if any two are in the same plane. From Figure 2 we conclude that the skew-symmetric matrix R_{An} is

$$\underline{R}_{An} = \begin{bmatrix} 0 & -L\sin\omega t_i & 0 \\ L\sin\omega t_i & 0 & -L\cos\omega t_i \\ 0 & L\cos\omega t_i & 0 \end{bmatrix} \tag{39}$$

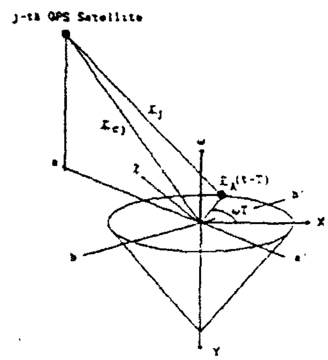

Figure 2. Geometry of Simplified Example

and thus, after some manipulation of trigonometric terms, we have

$$\Delta\underline{R}_{An} = \begin{bmatrix} 0 & -2L\sin\frac{\omega T}{2}\cos\omega(t_i-\frac{T}{2}) & 0 \\ -\text{sym} & 0 & 2L\sin\frac{\omega T}{2}\sin\omega(t_i-\frac{T}{2}) \\ 0 & -\text{sym} & 0 \end{bmatrix} \tag{40}$$

where "sym" denotes an entry equal to the symmetric entry in the matrix.

Since we have assumed that L is small compared to $\underline{r}_j$, $\underline{e}_{cj}$ is approximately equal to $\underline{e}_j$. Finally, assuming we are only interested in the y-component of the attitude error, we obtain from (37)

$$h_{j\delta\phi_y} = 2L\sin(\frac{\omega T}{2})\,[e_{cjx}\cos\omega(t_i-\frac{T}{2}) + e_{cjz}\sin\omega(t_i-\frac{T}{2})] \tag{41}$$

From (41) we conclude that $h_{j\delta\phi_y}$ is zero if any of the following three conditions occur

$$L = 0 \tag{42}$$

$$\frac{\omega T}{2} = N\pi \quad (N = 0, 1, 2, \ldots) \tag{43}$$

$$e_{cjx}\cos(\omega t_i-\frac{\omega T}{2}) + e_{cjz}\sin(\omega t_i-\frac{\omega T}{2}) = 0 \tag{44}$$

Obviously, the condition in (42) can occur if and only if the lever arm is zero. The condition in (43) can occur if there is no rotation or if the rotation rate is such that a multiple of 360 degrees is swept during the DR integration time T. The condition in (44) can occur if

$$e_{cjx} = e_{cjz} = 0 \tag{45}$$

or

$$\tan(\omega t_i-\frac{\omega T}{2}) = -\frac{e_{cjx}}{e_{cjz}} \tag{46}$$

The condition in (45) occurs if the satellite is directly above (or below) the vehicle; i.e., if the vector $\underline{r}_{cj}$ in Figure 2 has only a Y component. The condition in (46) occurs if the angle swept by the antenna lever arm during the DR integration interval is symmetrically distributed about the line bb' in the X-Z plane - see Figure 2.

We can express (41) in terms of the inner product between $\underline{e}_{cj}$ and the antenna location vector at the mid point of the DR integration, $\underline{r}_A(t_i\text{-}T/2)$.

$$\begin{aligned} h_{j\delta\phi_y} &= 2\sin(\frac{\omega T}{2})\,[\underline{e}_{cj}^{T}\,\underline{r}_A(t_i-\frac{T}{2})] \\ &= 2L\sin(\frac{\omega T}{2})\cos(\beta[\underline{e}_{cj},\underline{r}_A(t_i-\frac{T}{2})]) \end{aligned} \tag{47}$$

where $\beta[\underline{e}_{cj}, \underline{r}_A(t_i\text{-}T/2)]$ is the angle between the unit vector $\underline{e}_{cj}$ and the antenna location vector $\underline{r}_A(t_i\text{-}T/2)$.

From (47) we conclude that $h_{j\delta\phi_y}$ is sinusoidal and has a maximum peak value of 2L. This value is obtained when the vectors $\underline{e}_{cj}$ and $\underline{r}_A(t_i\text{-}T/2)$ are parallel and the antenna rotates 180 degrees starting and ending on line bb'. $h_{j\delta\phi_y}$ is zero if the vectors $\underline{e}_{cj}$ and $\underline{r}_A(t_i\text{-}T/2)$ are orthogonal.

Suppose now an Information Kalman Filter is used to estimate the constant attitude error $\delta\phi_y$ using DR measurements. The variance of the estimation error at t_k is (see [3])

$$(\sigma^2_{\epsilon\delta\phi_y}(t_k))^{-1} = (\sigma^2_{\epsilon\delta\phi_y}(t_0))^{-1} + \sum_{i=0}^{k} \frac{h_{j\delta\phi_y}(t_i)^2}{\sigma^2_{DR}} \quad (48)$$

where σ_{DR} denotes the sigma value of the observation noise. For sufficiently large t_k, we can assume that the sum can be evaluated using an average (and therefore constant) value in place of $h_{j\delta\phi y}(t_i)$. Then assuming the average value to be equal to half the peak value, we can show that for $t_k >> 0$, (see [1] for details)

$$\sigma_{\epsilon\delta\phi_y}(t_k) \leq \frac{\sigma_{DR}}{L\sin(\frac{\omega T}{2})\cos\beta} \quad (49)$$

We note that (49) is only a bound derived under very specific and limiting assumptions. Consequently, it is not very useful in determining attitude uncertainty bounds in more general situations. It does, however, help in revealing the general dependence of the attitude uncertainty to L and ωT. The following conclusions can now be drawn:

1) The uncertainty in estimating attitude errors $\sigma_{\epsilon\delta\phi}$ with an offset antenna is proportional to the DR measurement noise, i.e., σ_{DR}. (σ_{DR} is inversely proportional to the SNR at the input of the carrier tracking loop).

2) $\sigma_{\epsilon\delta\phi}$ is inversely proportional to the length of the antenna lever arm L.

3) $\sigma_{\epsilon\delta\phi}$ is minimized if the angle swept during the DR integration is 180 degrees.

4) $\sigma_{\epsilon\delta\phi}$ is minimized if the satellite is in the plane defined by the rotating antenna and the angle swept during the DR integration is symmetrically distributed about the vector r_{cj}.

5.0 SIMULATION RESULTS

This section summarizes covariance simulation analysis results for a spinning body using the more general results of section 3. Detailed discussion of this analysis is given in [1]. The scenario consists of a vehicle moving in an exoatmospheric trajectory (free fall) and spinning at a constant rate ω about its Y body frame axis. Figure 3 shows the vehicle's altitude profile. The vehicle is equipped with an integrated GPS/Strapdown Inertial navigator. The GPS antenna is mounted on the perimeter of the vehicle at a distance L from the Y_b axis. Therefore, the antenna rotates at the vehicle spin rate ω. To an

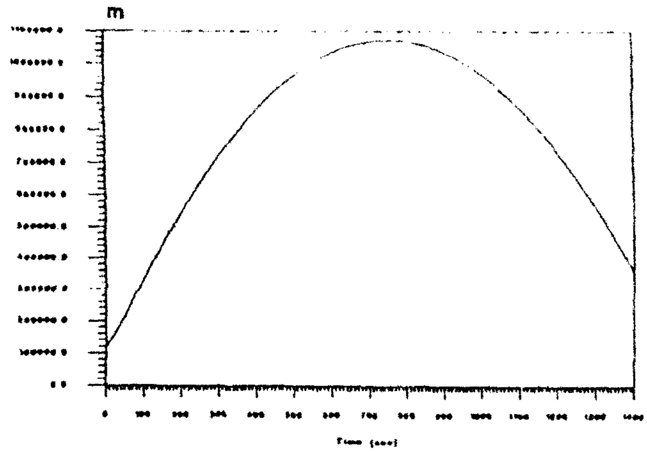

Figure 3. Altitude vs. Time

approximation, the scenario here resembles the conditions assumed in the analysis of section 4. Consequently, the insights gained in section 4 carry over to the more realistic conditions of this section. Typical errors were assumed for both the GPS and the INS (see [1] for details).

Results were obtained for a spin rate of ω=450 deg/sec and a DR integration time T=1 sec. Assuming first that the antenna was mounted on the spin axis (i.e. L=0), the RSS attitude error was shown to grow to about 30 degrees as a result of a 50 ppm scale factor error (see Figure 4). The inability to estimate this attitude error was the direct consequence of zero specific forces in free fall and therefore decoupling of attitude errors from velocity errors. When, under the same conditions, the antenna lever arm was set to L=7 inches, the RSS attitude error was estimated to 0.2 degrees. Furthermore, the scale factor error was now estimated to within a fraction of a ppm (see Figure 5).

Figures 6 and 7 show the sensitivity of the final attitude error to the lever arm length and spin rate, respectively, for T=0.78 sec. These results agree with the predictions in Section 4 (i.e. an inverse dependence on L and cosecant dependence on ω).

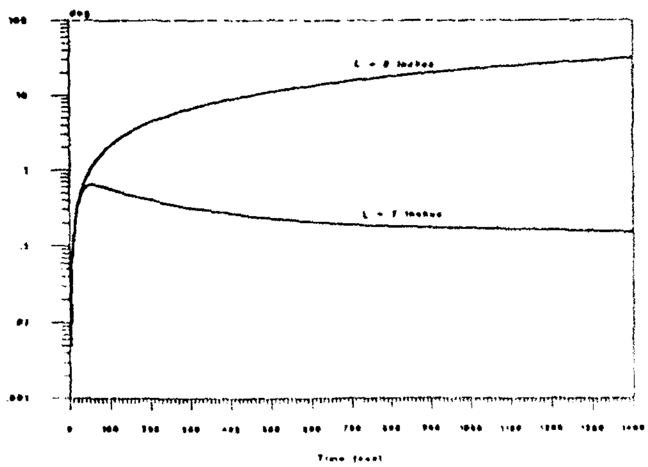

Figure 4 RSS Attitude Error Profile

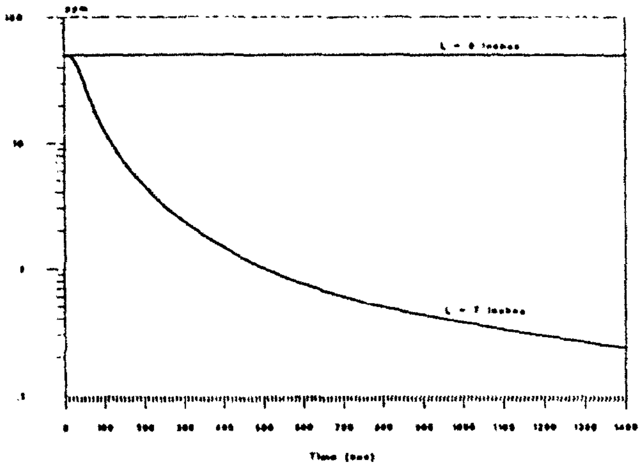

Figure 5. Y-Axis Gyro Scale Error Profile

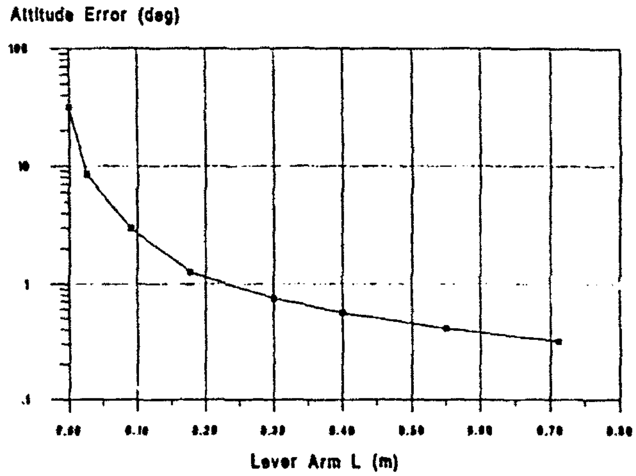

Figure 6. Final RSS Attitude Error Sensitivity to Antenna Lever Arm (T=0.78 sec, L=7 in)

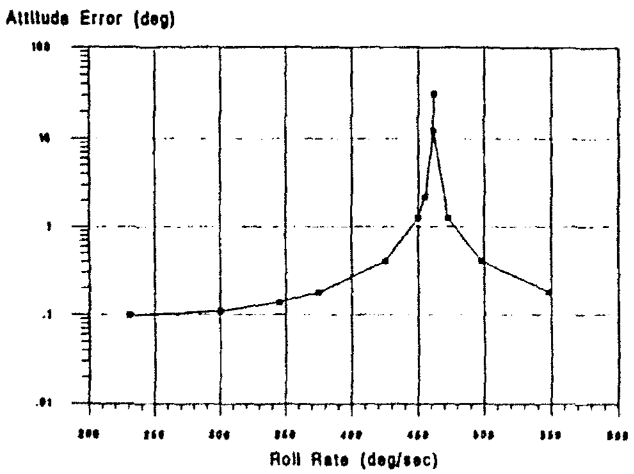

Figure 7. Final RSS Attitude Error Sensitivity to Lever Arm (T=.078 sec, L=7 in)

6.0 CONCLUSIONS

The method presented here appears promising as a GPS-based attitude estimation technique. It imposes minor implementation impacts but requires host vehicle attitude changes. Consequently, its main utility is in space applications. For terrestrial applications some benefit may be derived when rotary motion is inherently present, e.g., rotary wing aircraft.

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